Substitutions in Multiple Integrals

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Overview

In the lecture, we discuss how to evaluate multiple integrals by substitution.

As in single integration, the goal of substitution is to replace complicated integrals by ones that are easier to evaluate.

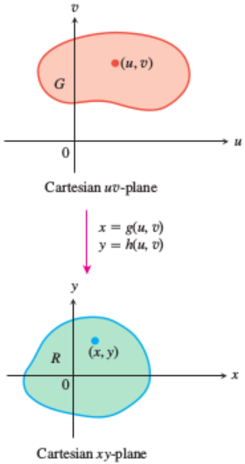
Substitutions accomplish this by simplifying the integrad, the limits of integration, or both.

Substitutions in Double Integrals

The polar coordinate substitution is a special case of a more general substitution method for double integrals, a method that pictures changes in variables as transformations of regions.

Suppose that a region G in the uv-plane is transformed one-to-one into the region R in the xy-plane by equations of the form

$$x = g(u, v), \quad y = h(u, v).$$



We call R the **image** of G under the transformation, and G the **preimage** of R. Any function $f(x, y)$ defined on R can be thought of as a function $f(g(u, v), h(u, v))$ defined on G as well.

How is the integral of $f(x, y)$ over R related to the integral of $g(g(u, v), h(u, v))$ over G?

The answer is : If g, h, and f have continuous partial derivatives and $J(u, v)$ (to be discussed in a moment) is zero only at isolated points, if at all, then

$$\iint\limits_R f(x, y)\ dx\ dy = \iint\limits_G f(f(u, v), h(u, v))\ |J(u, v)|\ du\ dv.$$

The above derivation is intricate and properly belongs to a course in advanced calculus. We do not give the derivation here.

Notice the "Reversed" Order

The transforming equations

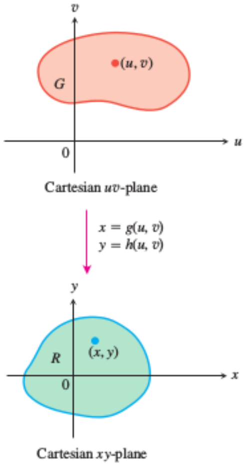
$$x = g(u, v) \quad \text{and} \quad y = h(u, v)$$

go from G to R, but **we use them to change an integral over R into an integral over G.**

Thus the equations

$$x = g(u, v) \quad \text{and} \quad y = h(u, v)$$

allow us to change an integral over a region R in the xy-plane into an integral over a region to G in the uv-plane.



Substitutions in Double Integrals

The factor $J(u, v)$ is the **Jacobian** of the coordinate transformation, named after German mathematician Carl Jacobi.

It measures how much the transformation is expanding or contracting the area around a point in G as G is transformed into R.

Carl Gustav Jacob Jacobi

(1804-1851)

Definition 1.

The Jacobian determinant *or* Jacobian *of the coordinate transformation* $x = g(u, v),\ y = h(u, v)$ *is*

$$J(u, v) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial u} \end{vmatrix}$$

The Jacobian is also denoted by $J(u, v) = \frac{\partial(x,y)}{\partial(u,v)}$ to help remember how the determinant is constructed from the partial derivatives of x and y.

Remember : As we are going from a old set of variables "x and y" to a new set of variables "u and v", x and y are to be integrated partially with respect to u and v.

Substitutions in Double Integrals - An Example

For polar coordinates, we have r and θ in place of u and v.

The transformation from Cartesian $r\theta$-plane (not in polar coordinate plane) to Cartesian xy-plane is given by $x = r\cos\theta$ and $y = r\sin\theta$. The Jacobian of the transformation is

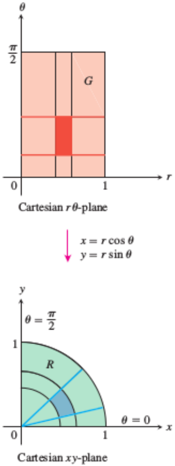
$$J(r, \theta) = r$$

and hence

$$\iint\limits_R f(x, y)\ dx\ dy = \iint\limits_G f(r\cos\theta, r\sin\theta)\ |r|\ dr\ d\theta.$$

Substitutions in Double Integrals - An Example

We can drop the absolute value signs whenever $r \geq 0$.



Substitutions in Double Integrals - An Example

The equations $x = r\cos\theta, y = r\sin\theta$ transform the rectangle

$$G : 0 \leq r \leq 1, \quad 0 \leq \theta \leq \pi/2$$

into the quarter circle R bounded by

$$x^2 + y^2 = 1$$

in the first quadrant of the xy-plane.

Notice that the integral on the right-hand side of the above equation is not the integral of $f(r\cos\theta, r\sin\theta)$ over a region in the polar coordinate plane.

It is the integral of the product of $f(r\cos\theta, r\sin\theta)$ and r over a region G in the Cartesian $r\theta$-plane.

To evaluate

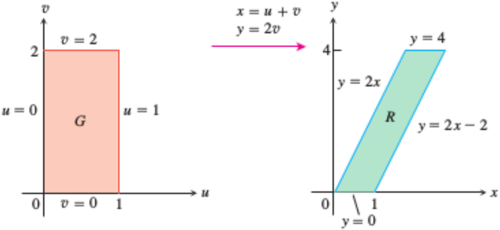
$$\int_0^4 \int_{x=y/2}^{x=(y/2)+1} \frac{2x-y}{2}\, dx\, dy$$

by applying the transformation

$$u = \frac{2x-y}{2}, \quad v = \frac{y}{2},$$

and integrating over an appropriate region in the uv-plane.

We sketch the region R of integration in the xy-plane and identify its boundaries.



Substitutions in Double Integrals - An Example

We first need to find the corresponding uv-region G and the Jacobian of the transformation.

Since $x = u + v$ and $y = 2v$, we then find the boundaries of G by substituting these expressions into the equations for the boundaries of R.

The Jacobian of the transformation is $J(u, v) = 1$.

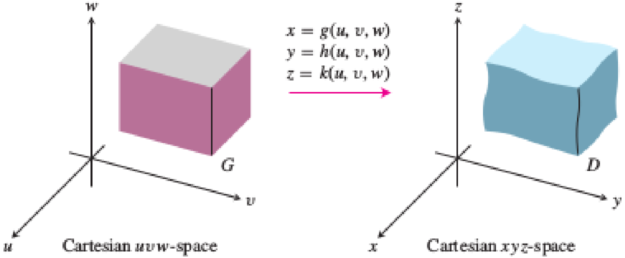
Therefore

$$\begin{aligned}
\int_0^4 \int_{x=y/2}^{x=(y/2)+1} \frac{2x-y}{2}\, dx\, dy &= \int_{v=0}^{v=2} \int_{u=0}^{u=1} u\, |J(u,v)|\, du\, dv \\
&= \int_{v=0}^{v=2} \int_{u=0}^{u=1} (u)\,(2)\, du\, dv = 2.
\end{aligned}$$

Substitutions in Triple Integrals

Suppose that a region G in the *uvw*-space is transformed one-to-one into the region D in *xyz*-space by differentiable equations of the form

$$x = g(u, v, w), \quad y = h(u, v, w), \quad z = k(u, v, w).$$



Substitutions in Triple Integrals

Then any function $F(x, y, z)$ defined on D can be thought of as a function

$$F(g(u, v, w), h(u, v, w), k(u, v, w)) = H(u, v, w)$$

defined on G.

If g, h, and k have continuous first partial derivatives, then the integral $F(x, y, z)$ over D is related to the integral of $H(u, v, w)$ over G bt the equation

$$\iiint\limits_D F(x, y, z)\ dx\ dy\ dz = \iiint\limits_G H(u, v, w)\ |J(u, v, w)|\ du\ dv\ dw.$$

The above derivation is intricate and properly belongs to a course in advanced calculus. We do not give the derivation here.

Substitutions in Triple Integrals

The factor $J(u, v, w)$ is the **Jacobian determinant**

$$J(u,v,w) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial u} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial u} & \frac{\partial z}{\partial w} \end{vmatrix} = \frac{\partial(x,y,z)}{\partial(u,v,w)}.$$

This determinant measures how much the volume near a point in G is being expanded or contracted by the transformation from (u, v, w) to (x, y, z) coordinates.

Substitutions in Triple Integrals - An Example

For cylindrical coordinates, r, θ, and z take the place of u, v, and w.

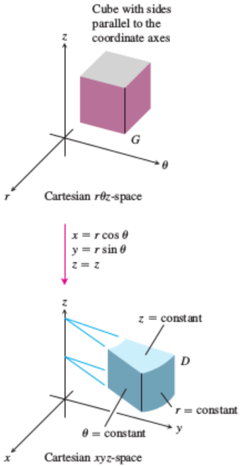
The transformation from Cartesian $r\theta z$-space to Cartesian xyz-space is given by the equations

$$x = r\cos\theta \quad \text{and} \quad y = r\sin\theta, \quad z = z.$$

The Jacobian of the transformation is $J(r,\theta,z) = r$ and hence

$$\iiint_D F(x,y,z)\,dx\,dy\,dz = \iiint_G H(r\cos\theta, r\sin\theta, z)\,|r|\,dr\,d\theta\,dz.$$

We can drop the absolute value signs whenever $r \geq 0$.



Substitutions in Triple Integrals - An Example

For spherical coordinates, ρ, ϕ, and θ take the place of u, v, and w. The transformation from Cartesian $\rho\phi\theta$-space to Cartesian xyz-space is given by the equations

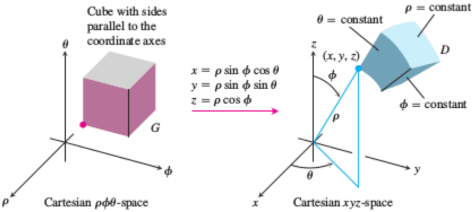
$$x = \rho \sin\phi \cos\theta \qquad y = \rho \sin\phi \sin\theta, \quad z = \rho \cos\phi.$$

The Jacobian of the transformation is $J(r, \phi, \theta) = \rho^2\ \sin\phi$ and hence

$$\iiint_D F(x, y, z)\ dx\ dy\ dz = \iiint_G H(r\phi, \theta)\ |\rho^2\ \sin\phi|\ d\rho\ d\phi\ d\theta.$$

Substitutions in Triple Integrals - An Example

We can drop the absolute value signs because $\sin\phi$ is never negative for $0 \leq \phi \leq \pi$.



Substitutions in Triple Integrals - An Example

To evaluate

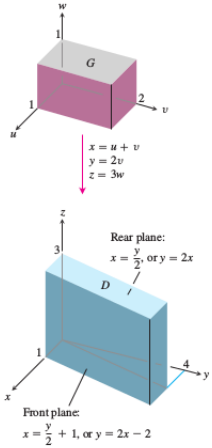
$$\int_0^3 \int_0^4 \int_{x=y/2}^{x=(y/2)+1} \left(\frac{2x-y}{2} + \frac{z}{3}\right) dx\, dy\, dz$$

by applying the transformation

$$u = \frac{2x-y}{2}, \quad v = \frac{y}{2}, \quad w = \frac{z}{3}$$

and integrating over an appropriate region in the *uvw*-space.

We sketch the region D of integration in the *xyz*-space and identify its boundaries. In this case, the bounding surfaces are planes.



We first need to find the corresponding *uvw*-region G and the Jacobian of the transformation.

Since $x = u + v, \quad y = 2v, \quad z = 3w$, we then find the boundaries of G by substituting these expressions into the equations for the boundaries of D.

The Jacobian of the transformation is $J(u, v, w) = 6$.

Therefore

$$\begin{aligned}
\int_0^3 \int_0^4 \int_{x=y/2}^{x=(y/2)+1} \left(\frac{2x-y}{2} + \frac{z}{3}\right) dx\ dy\ dz &= \int_0^1 \int_0^2 \int_0^1 (u+w)\ |J(u,v,w)|\ du\ dv\ dw \\
&= \int_0^1 \int_0^2 \int_0^1 (u+w)\ (6)\ du\ dv\ dw \\
&= 12.
\end{aligned}$$

Exercise 2.

(a) *Solve the system*

$$u = 3x + 2y, \quad v = x + 4y$$

for x *and* y *in terms of* u *and* v. *Then find the value of the Jacobian* $\partial(x, y)/\partial(u, v)$.

(b) *Find the image under the transformation*

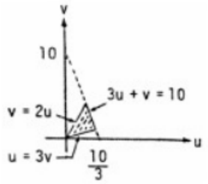
$$u = 3x + 2y, \quad v = x + 4y$$

of the triangular region in the xy*-plane bounded by the* x*-axis, the* y*-axis, and the line* $x + y = 1$. *Sketch the transformed region in the* uv*-plane.*

Solution for Exercise 2

(a) $3x + 2y = u$ and $x + 4y = v \Rightarrow -5x = -2u + v$ and $y = \frac{1}{2}(u - 3x) \Rightarrow x = \frac{1}{5}(2u - v)$ and $y = \frac{1}{10}(3v - u)$;
$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{2}{5} & -\frac{1}{5} \\ -\frac{1}{10} & \frac{3}{10} \end{vmatrix} = \frac{6}{50} - \frac{1}{50} = \frac{1}{10}$.

(b) The x-axis $y = 0 \Rightarrow u = 3v$; the y-axis $x = 0$
$\Rightarrow v = 2u$; the line $x + y = 1$
$\Rightarrow \frac{1}{5}(2u - v) + \frac{1}{10}(3v - u) = 1$
$\Rightarrow 2(2u - v) + (3v - u) = 10 \Rightarrow 3u + v = 10.$
The transformed region is shown below.



Exercise 3.

Use the transformation

$$u = 3x + 2y, \quad v = x + 4y$$

to evaluate the integral

$$\iint_R (3x^2 + 14xy + 8y^2)\ dx\ dy$$

for the region R *in the first quadrant bounded by the lines* $y = -(3/2)x + 1$, $y = -(3/2)x + 3$, $y = -(1/4)x$, *and* $y = -(1/4)x + 1$.

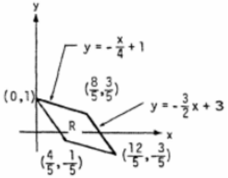
Solution for Exercise 3

$$\iint_R (3x^2 + 14xy + 8y^2) dx\, dy$$
$$= \iint_R (3x + 2y)(x + 4y)\, dx\, dy$$
$$= \iint_G uv \left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\, dv = \frac{1}{10} \iint_G uv\, du\, dv;$$

We find the boundaries of G from the boundaries of R, shown in the accompanying figure:

xy-eqns. for boundary of R	Corresponding uv-eqns. for boundary of G	Simplified uv-eqns.
$y = -\frac{3}{2}x + 1$	$\frac{1}{10}(3v - u) = -\frac{3}{10}(2u - v) + 1$	$u = 2$
$y = -\frac{3}{2}x + 3$	$\frac{1}{10}(3v - u) = -\frac{3}{10}(2u - v) + 3$	$u = 6$
$y = -\frac{1}{4}x$	$\frac{1}{10}(3v - u) = -\frac{1}{20}(2u - v)$	$v = 0$
$y = -\frac{1}{4}x + 1$	$\frac{1}{10}(3v - u) = -\frac{1}{20}(2u - v) + 1$	$v = 4$

$$\Rightarrow \frac{1}{10} \iint_G uv\, du\, dv = \frac{1}{10} \int_2^6 \int_0^4 uv\, dv\, du = \frac{1}{10} \int_2^6 u \left[\frac{v^2}{2}\right]_0^4 du = \frac{4}{5} \int_2^6 u\, du = \left(\frac{4}{5}\right) \left[\frac{u^2}{2}\right]_2^6 =$$
$$\left(\frac{4}{5}\right)(18 - 2) = \frac{64}{5}$$



Exercise 4.

Let R be the region in the first quadrant of the xy-plane bounded by the hyperbolas $xy = 1$, $xy = 9$ and the lines $y = x$, $y = 4x$. Use the transformation $x = u/v$, $y = uv$ with $u > 0$ and $v > 0$ to rewrite

$$\iint_R \left(\sqrt{\frac{x}{y}} + \sqrt{xy} \right) \, dx \, dy$$

as an integral over an appropriate region G in the uv-plane. Then evaluate the uv-integral over G.

Solution for Exercise 4

$x = \frac{u}{v}$ and $y = uv \Rightarrow \frac{y}{x} = v^2$ and $xy = u^2$;
$\frac{\partial(x,y)}{\partial(u,v)} = J(u,v) = \begin{vmatrix} v^{-1} & -uv^{-2} \\ v & u \end{vmatrix} = v^{-1}u + v^{-1}u = \frac{2u}{v}$; $y = x \Rightarrow uv = \frac{u}{v} \Rightarrow v = 1$, and
$y = 4x \Rightarrow v = 2$; $xy = 1 \Rightarrow u = 1$, and $xy = 9 \Rightarrow u = 3$; thus
$\iint\limits_R \left(\sqrt{\frac{y}{x}} + \sqrt{xy}\right) dx\, dy = \int_1^3 \int_1^2 (v+u)\left(\frac{2u}{v}\right)\, dv\, du = \int_1^3 \int_1^2 \left(2u + \frac{2u^2}{v}\right) dv\, du =$
$\int_1^3 \left[2uv + 2u^2 \ln v\right]_1^2 du = \int_1^3 (2u + 2u^2 \ln 2) du = \left[u^2 + \frac{2}{3}u^2 \ln 2\right]_1^3 = 8 + \frac{2}{3}(26)(\ln 2) = 8 + \frac{52}{3}(\ln 2)$

Exercise

Exercise 5.

(a) *Find the Jacobian of the transformation* $x = u, y = uv$ *and sketch the region* $G : 1 \leq u \leq 2, 1 \leq uv \leq 2$, *in the uv-plane.*

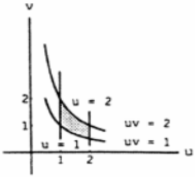
(b) *Transform the integral*

$$\int_1^2 \int_1^2 \frac{y}{x} dy \; dx$$

into an equivalent integral.

Solution for Exercise 5

(a) $\frac{\partial(x,y)}{\partial(u,v)} = J(u,v) = \begin{vmatrix} 1 & 0 \\ v & u \end{vmatrix} = u$, and the region G is shown below.



(b)

$$\int_1^2 \int_1^2 \frac{y}{x} dy\ dx = \int_1^2 \int_{1/u}^{2/u} uv\ dv\ du = \frac{3}{2} \ln 2.$$

The area of an ellipse

Exercise 6.

The area πab of the ellipse

$$x^2/a^2 + y^2/b^2 = 1$$

can be found by integrating the function $f(x, y) = 1$ over the region bounded by the ellipse in the $xy-$plane.
Evaluating the integral directly requires a trigonometric substitution. An easier way to evaluate the integral is to use the transformation

$$x = au, \quad y = bv$$

and evaluate the transformed integral over the disk $G : u^2 + v^2 \leq 1$ in the $uv-$plane. Find the area this way.

Solution for Exercise 6

$\frac{\partial(x,y)}{\partial(u,v)} = J(u,v) = \begin{vmatrix} a & 0 \\ 0 & b \end{vmatrix} = ab;$

$A = \iint\limits_R dy\,dx = \iint\limits_G ab\,du\,dv = \int_{-1}^{1} \int_{-\sqrt{1-u^2}}^{\sqrt{1-u^2}} ab\,dv\,du =$

$2ab \int_{-1}^{1} \sqrt{1-u^2}du = 2ab\left[\frac{u}{2}\sqrt{1-u^2} + \frac{1}{2}\sin^{-1}u\right]_{-1}^{1} =$

$ab[\sin^{-1}1 - \sin^{-1}(-1)] = ab\left[\frac{\pi}{2} - \left(-\frac{\pi}{2}\right)\right] = ab\pi$

Exercise

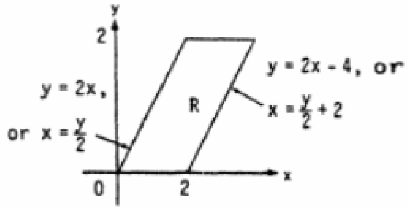
Exercise 7.

Use the transformation $x = u + (1/2)v, y = v$ *to evaluate the integral*

$$\int_0^2 \int_{y/2}^{(y+4)/2} y^3(2x - y)e^{(2x-y)^2} dx\, dy$$

by first writing it as, an integral over a region G *in the* uv*-plane.*

Solution for Exercise 7



$x = u + \frac{v}{2}$ and $y = v \Rightarrow 2x - y = (2u + v) - v = 2u$ and

$\frac{\partial(x,y)}{\partial(u,v)} = J(u, v) = \begin{vmatrix} 1 & \frac{1}{2} \\ 0 & 1 \end{vmatrix} = 1;$

Solution for Exercise 7 (contd...)

Next, $u = x - \frac{v}{2} = x - \frac{y}{2}$ and $v = y$, so the boundaries of the region of integration R in the xy-plane are transformed to the boundaries of G:

xy-eqns.	Corresponding uv-eqns.	Simplified uv-equations
$x = \frac{y}{2}$	$u + \frac{v}{2} = \frac{v}{2}$	$u = 0$
$x = \frac{y}{2} + 2$	$u + \frac{v}{2} = \frac{v}{2} + 2$	$u = 2$
$y = 0$	$v = 0$	$v = 0$
$y = 2$	$v = 2$	$v = 2$

$$\Rightarrow \int_0^2 \int_{y/2}^{(y/2)+2} y^3(2x - y)e^{(2x-y)^2}\,dx\,dy = \int_0^2 \int_0^2 v^3(2u)e^{4u^2}\,du\,dv =$$
$$\int_0^2 v^3 \left[\frac{1}{4}e^{4u^2}\right]_0^2 dv = \frac{1}{4}\int_0^2 v^3(e^{16} - 1)dv = \frac{1}{4}(e^{16} - 1)\left[\frac{v^4}{4}\right]_0^2 = e^{16} - 1$$

Exercise

Exercise 8.

Use the transformation $x = u/v, y = uv$ *to evaluate the integral sum*

$$\int_1^2 \int_{1/y}^{y} (x^2 + y^2) dx\ dy + \int_2^4 \int_{y/4}^{4/y} (x^2 + y^2) dx\ dy.$$

Solution for Exercise 8

$x = \frac{u}{v}$ and $y = uv \Rightarrow \frac{y}{x} = v^2$ and $xy = u^2$;

$\frac{\partial(x,y)}{\partial(u,v)} = J(u,v) = \begin{vmatrix} v^{-1} & -uv^{-2} \\ v & u \end{vmatrix} = v^{-1}u + v^{-1}u = \frac{2u}{v}$;

$y = x \Rightarrow uv = \frac{u}{v} \Rightarrow = 2$; thus $\int_1^2 \int_{1/y}^{y} (x^2 + y^2) dx\, dy =$

$\int_1^2 \left[\frac{u^4}{2v^3} + \frac{1}{2} u^4 v \right]_1^2 dv = \int_1^2 \left(\frac{15}{2v^3} + \frac{15v}{2} \right) dv = \left[-\frac{15}{4v^2} + \frac{15v^2}{4} \right]_1^2 = \frac{225}{16}$.

Exercise 9.

Find the Jacobian $\delta(x, y, z)/\delta(u, v, w)$ *of the transformation*

(a) $x = u\cos v, \quad y = u\sin v, \quad z = w$

(b) $x = 2u - 1, \quad y = 3v - 4, \quad z = (1/2)(w - 4).$

Solution for Exercise 9

(a) $x = u\cos v,\ y = u\sin v,$

$$z = w \Rightarrow \frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} \cos v & -u\sin v & 0 \\ \sin v & u\cos v & 0 \\ 0 & 0 & 1 \end{vmatrix} = u\cos^2 v + u\sin^2 v = u$$

(b) $x = 2u - 1,\ y = 3v - 4,$

$$z = \tfrac{1}{2}(w-4) \Rightarrow \frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & \frac{1}{2} \end{vmatrix} = (2)(3)(\tfrac{1}{2}) = 3$$

Exercise 10.

How can substitutions in single definite integrals be viewed as transformations of regions? What is the Jacobian in such a case? Illustrate with an example.

Solution for Exercise 10

Let $u = g(x) \Rightarrow J(x) = \frac{du}{dx} = g'(x)$.

Then $\displaystyle\int_a^b f(u)du = \int_{g(a)}^{g(b)} f(g(x))g'(x)dx$.

Note that $g'(x)$ represents the Jacobian of the transformation

$$u = g(x) \quad \text{or} \quad x = g^{-1}(u).$$

Exercise

Exercise 11.

Find the volume of the ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1,$$

using a suitable change of variables.

Solution for Exercise 11

Let $x = au, y = bv$, and $z = vw$.

$J(u, v, w) = \begin{vmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{vmatrix} = abc$; the transformation takes the ellipsoid region

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} \leq 1$$

in xyz-space into the spherical region

$$u^2 + v^2 + w^2 \leq 1$$

in uvw-space (which has volume $V = \frac{4}{3}\pi$).

Then $V = \iiint\limits_R dx\, dy\, dz = \iiint\limits_G abc\, du\, dv\, dw = \frac{4\pi abc}{3}$.

Exercise 12.

Evaluate

$$\iiint |xyz| \; dx \; dy \; dz$$

over the solid ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} \leq 1,$$

using a suitable substitution.

Solution for Exercise 12

Let $x = au, y = bv$, and $z = vw$.

$J(u, v, w) = \begin{vmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{vmatrix} = abc$; the transformation takes the ellipsoid region

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} \leq 1$$

in xyz-space into the spherical region $G : u^2 + v^2 + w^2 \leq 1$ in uvw-space. Thus

$$\iiint |xyz| \; dx \; dy \; dz = \iiint_G a^2 b^2 c^2 \; uvw \; dw \; dv \; du.$$

Applying spherical coordinate system, we get $V = \frac{a^2 b^2 c^2}{6}$.

Exercise 13.

Let D be the region in xyz-space defined by the inequalities

$$1 \leq x \leq 2, \quad 0 \leq xy \leq 2, \quad 0 \leq z \leq 1.$$

Evaluate

$$\iiint_D (x^2y + 3xyz)\ dx\ dy\ dz$$

by applying the transformation

$$u = x, \quad v = xy, \quad w = 3z$$

and integrating over an appropriate region G in uvw-space.

Solution for Exercise 13

Let $u = x, v = xy$, and $w = z$. Then $J(u, v, w) = \frac{1}{u}$.

Thus

$$\iiint\limits_D (x^2 y + 3xyz)\; dx\; dy\; dz = \int_0^1 \int_0^2 \int_1^2 \frac{uv + 3vw}{u}\; du\; dv\; dw = 2 + \ln 8.$$

References

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